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THE DYNAMICS OF SLENDER BODIES IN DENSE MEDIA UNDER CONDITIONS OF THE LOCAL INTERACTION MODEL†

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Using a model of local interactions an asymptotic theory is constructed for the plane motion of a slender sharpened body of revolution when it is fully immersed in a dense medium and there is no flow separation at the lateral surface. Assuming that at the initial instant of time the dynamic component of the normal stress is much greater than the tensile strength, the domain of variation of the governing parameters corresponding to asymptotically stable types of motion is found for translational motion of a body at zero angle of attack.

The difficulties in computing the motion of a rigid body in a dense medium, such as various types of soil, are caused, in particular, by the unsteady-state nature of the problem, the variation of the velocity of the centre of mass of the body over a wide range compared with the characteristic velocity of propagation of perturbations in the medium, the possibility of modes of motion with flow separation at the body surface, etc. Therefore, in many cases in place of the joint problem of the motion of the body and the medium one considers, for simplicity, the motion of a body subject to a force and moment computed using local interaction models contained in the known approximate solutions of penetration problems or determined empirically [1]. However, in this simplification the solution of the Cauchy problem can be obtained only by integrating the system of equations of motion of a rigid body numerically, while the analysis of the stability of the motion as a function of the governing parameters and initial conditions is a complex problem.

Below we use a two-term model of local interactions to compute the pressure on the contact surface between the body and the medium. The model contains dynamic term proportional to the square of the total velocity of the point on the surface of the body under consideration and a constant term characterizing the deformation resistance capacity of the medium. Tangential stresses on the contact surface are determined using Coulomb's model of friction under normal stresses of arbitrary magnitude. This enables us to estimate the maximum possible effect of friction on the body dynamics. The assumptions that the body of revolution is a slender one and the dynamic term being the dominant one in the normal stress model at the initial instant of time are crucial for obtaining an analytic solution of the problem and a criterion for the motion to be asymptotically stable, that is valid for arbitrary initial values of the special phase variables in a domain corresponding to flow without separation. Bearing in mind the effect of friction, stabilizing the result obtained using Coulomb's model of friction can be regarded as an "upper limit" of the boundary of the stability domain for a slender body, while the result obtained for a dry coefficient friction of zero can be regarded as a "lower limit".

1. THE INTERACTION MODEL. FORCE AND MOMENT

In the two-term model of local interactions the normal stress on the contact surface between the body and the medium is given by

$$\sigma_n = A(\mathbf{v}\mathbf{n})^2 + C > 0, \quad \mathbf{v} = \mathbf{v}_m + [\boldsymbol{\omega} \times \mathbf{r}] \tag{1.1}$$

Here $\mathbf{v}_m$ and $\mathbf{v}$ are the velocities of the centre of mass and of the body surface at the point under consideration, $\mathbf{n}$ is the outward unit normal vector to the body surface, A and C are constant parameters of the local model, which depend on the characteristics of the medium according to formulae obtained in approximate theories or are experimental constants [1], $\boldsymbol{\omega}$ is the angular velocity vector of the body

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having one non-zero component ω in the case of plane motion and r is the radius vector drawn from the centre of mass of the body to a point on its surface.

Unlike the motion of a body in a gas, where the main terms of the drag acting on an elementary surface element ΔS of a slender body are given by $\Delta f \sim q(c_p\beta + c_f)\Delta S$ (c_p and c_f are the pressure coefficient and the coefficient of friction, β is a number proportional to the relative half-thickness of the body if the inclination of the generatrix of the body of revolution varies only slightly, and q is the velocity head), where the coefficient of friction can be neglected in many cases, the motion of a body in a dense medium in Coulomb's model of friction on the contact surface, in which case $\Delta f \sim \sigma_n(\beta + \mu)\Delta S$ (μ being the coefficient of friction) is affected by friction, the contribution of which to the resistance is of the same order as the pressure and cannot be neglected. Therefore, using (1.1), we can write the expressions for the vectors of the force and moment about the centre of mass, which act on the moving body as follows:

$$\mathbf{F} = \iint_S \sigma_n(-\mathbf{n} + \mu\mathbf{t})dS \tag{1.2}$$

$$\mathbf{M} = \iint_S \sigma_n(-[\mathbf{r}\times\mathbf{n}] + \mu[\mathbf{r}\times\mathbf{t}])dS \tag{1.3}$$

$$\mathbf{t} = [[\mathbf{v}\times\mathbf{n}]\times\mathbf{n}]/|[\mathbf{v}\times\mathbf{n}]| \tag{1.4}$$

Here $\mathbf{t}$ is a unit tangent vector to the body surface at the point under consideration, and in the general case S is the part of the body surface which is currently in contact with the medium. If the range of possible modes of motion is not limited and separation between the medium and the surface of the body is admitted along some unspecified curve surrounding the domain of integration $S = S(\upsilon_m, \alpha, \omega)$, where α is the angle of incidence, then the problem of the body dynamics cannot be solved analytically. We assume that the whole lateral surface of the body remains in contact with the medium. This imposes a restriction on the domain of possible variation of the initial and actual values of υ_m, α and ω, which must satisfy the inequality

$$(\mathbf{v}\cdot\mathbf{n}) > 0 \tag{1.5}$$

when there is no flow separation within the framework of the local interaction model.

First we need to specify the shape of the body.

2. A THIN CONE. FORMULATION OF THE PROBLEM

We introduce two rectangular systems of coordinates with origin at the centre of mass of the body: (τ, n, z) attached to the body and with the τ axis coinciding with the axis of the cone and directed towards its vertex, and (υ, w, z), the velocity system in which the υ axis has the same direction as the vector υ_m. The coordinate planes (τ, n) and (υ, w) coincide with the plane of motion of the body. The angle of incidence α, that is, the angle between the υ and τ axes, will be assumed positive if the τ axis is rotated relative to the υ axis in an anticlockwise direction. The normal vector $\mathbf{n}$ to the surface of the cone with half-angle β at the vertex, the position vector $\mathbf{r}$ an the velocity υ in (1.1) in the attached system of coordinates can be written in the form (below all linear dimensions are relative to the height L of the cone)

$$\mathbf{n} = \{\sin\beta, \cos\beta\cos\varphi, \cos\beta\sin\varphi\} \tag{2.1}$$

$$\mathbf{r} = L\{\tau, R\cos\varphi, R\sin\varphi\} \tag{2.2}$$

$$R = (c_m - \tau)\operatorname{tg}\beta, \quad \tau\in[c_m - 1, c_m], \quad \varphi\in[0, 2\pi]$$

$$\mathbf{v} = \{v_m\cos\alpha - \omega RL\cos\varphi,\ -v_m\sin\alpha + \omega L\tau,\ 0\} \tag{2.3}$$

Here c_m is the relative distance between the vertex of the cone and its centre of mass, and (τ, R, φ) are cylindrical coordinates.

Substituting (2.1) and (2.3) into (1.5), we find that in the plane of variables

$$k = \frac{\operatorname{tg}\alpha}{\operatorname{tg}\beta}, \quad k_1 = \frac{\omega L}{v_m\sin\beta\cos\beta\cos\alpha} \tag{2.4}$$

which depend on the geometry of the body and on υ_m, α and ω, the domain of flow past the cone without separation is a parallelogram with one pair of opposite vertices lying on the axis $k_1 = 0$ ($k = \pm 1$) and the other pair on the straight lines $k_1 = \pm 2$ with

$$\begin{aligned} &-2 < k_1 \leqslant 0, \quad \left(c_m \cos^2\beta - 1\right)k_1 - 1 < k < c_m \cos^2\beta k_1 + 1 \\ &0 \leqslant k_1 < 2, \quad c_m \cos^2\beta k_1 - 1 < k < \left(c_m \cos^2\beta - 1\right)k_1 + 1 \end{aligned} \tag{2.5}$$

Since the problem of the plane motion of a cone is considered in the slender-body approximation ($\beta^2 \ll 1$), using the condition $|k| \leqslant 1$ (2.5) below we shall also neglect the terms α^2 compared to unity. In this approximation, using (2.1)–(2.5) and the differential relation $dS = (c_m - \tau)\beta L^2 d\tau d\varphi$, we can find the vector $\mathbf{t}$ (1.4) and also the components of the force vector (1.2) and moment vector (1.3) in the attached system of coordinates

$$\mathbf{t} = \left\{-1,\ \beta\left[\cos\varphi + (k - \tau k_1)\sin^2\varphi\right],\ \beta\sin\varphi\left[1 - (k - \tau k_1)\cos\varphi\right]\right\}$$

$$F_\tau = -\pi B(\beta + \mu)\left[1 + D + \frac{1}{2}K^2(k, k_1) + \frac{1}{36}k_1^2\right] \tag{2.6}$$

$$F_n = \pi B\left\{K\left[1 + \frac{1}{2}\beta\mu\left(D - 1 + \frac{1}{4}K^2 + \frac{1}{24}k_1^2\right)\right] - \frac{1}{1080}\beta\mu k_1^3\right\} \tag{2.7}$$

$$\begin{aligned} M_z = -\pi BL\Bigg\{ & z_y k + K_0 k_1 + \beta\mu\Bigg[\left(\frac{2}{3} + \frac{1}{2}(D-1)z_y\right)K + \frac{1}{36}(1+D)k_1 + \\ & + \frac{1}{8}\left(z_y K^3 + \frac{1}{6}K^2 k_1 - \frac{1}{6}\left(\frac{2}{15} - z_y\right)K k_1^2 + \frac{1}{135}\left(1 - z_y\right)k_1^3\right)\Bigg]\Bigg\} \end{aligned} \tag{2.8}$$

Here

$$\begin{aligned} &B = A\upsilon_m^2 L^2\beta^3, \quad D = C/\left(A\upsilon_m^2\beta^2\right), \quad z_y = 2/3 - c_m \\ &K = k + z_y k_1, \quad K_0 = 1/18 + z_y^2 \end{aligned} \tag{2.9}$$

The parameter D characterizes the relative contribution of the deformation and dynamic components to the pressure on the body surface (1.1) in translational motion at zero angle of incidence and z_y is the overall static stability of a cone in a gas in this approximation [2] using the quasisteady interaction model ($\omega = 0$). In the quasisteady approximation, from (2.7) and (2.8) we can find the overall static stability of a cone using the interaction model involving Coulomb friction

$$z_y^\tau = z_y + \frac{2}{3}\beta\mu\left[1 - \frac{1}{2}\beta\mu\left(1 - D - \frac{1}{4}k^2\right)\right]^{-1} \tag{2.10}$$

Expressions (2.6)–(2.8) written in the slender-body approximation can be simplified considerably with additional assumptions, which enables us to find an effective solution of the system of equations of motion of the rigid body.

Let $D_0 \equiv D(\upsilon_{m0}) = C/(A\upsilon_{m0}^2\beta^2) \ll 1$, which corresponds to the dominant role of the dynamic component in the interaction model at the initial instant of time. Note that under this condition one can neglect the loss of speech up to the stage when the body completely enters the medium [3]. Omitting the estimates, we observe that, with the above assumption, one can eliminate from the equations of motion the terms related to gravitation, which are significant only if the term characterizing the deformation resistance capacity of the medium begins to predominate in interaction model (1.1). Thus for soils $\mu \sim 0, 2$ we shall neglect the terms $O(\beta\mu)$ in (2.7) and (2.8) compared to unity, which in practice does not cause any deterioration in the slender body approximation. By (2.7), (2.8) and the expressions for the components $F_\upsilon = F_\tau - \alpha F_n$, $F_w = \alpha F_\tau + F_n$ of the force vector $\mathbf{F}$ in the velocity system of coordinates we should have D 1 in the discarded terms in order to preserve the accuracy of the adopted approximation. As will be seen below, the angular motion of the body about the centre of mass will practically disappear before D leaves the range $[D_0, 1]$ as the velocity decreases.

Using the above simplifications, we can write the system of equations of motion in the form

$$\dot{\upsilon}_m = -\frac{\upsilon_m^2\beta^2}{L}A_k\left\{\left(1+\frac{\mu}{\beta}\right)\left[1+D+\frac{1}{2}K^2+\frac{1}{36}k_1^2\right]+kK\right\} \equiv \frac{1}{M_0}F_\upsilon, \quad A_k = 3\frac{A}{\rho_0} \tag{2.11}$$

$$\dot{\theta} = \frac{\upsilon_m\beta}{L}A_kK \equiv \frac{1}{M_0\upsilon_m}F_w \tag{2.12}$$

$$\dot{\omega} = -\frac{\upsilon_m^2\beta}{IL^2}A_kK_0K_\omega \equiv \frac{1}{I_z}M_z \tag{2.13}$$

$$K_\omega = k_1 - p_\omega k, \quad p_\omega = -\frac{z_y^\mathrm{T}}{K_0}, \quad I = \frac{I_z}{M_0L^2}$$

$$\dot{k} = \frac{\upsilon_m}{L}K_1K_\alpha, \quad K_1 = 1 - z_yA_k, \quad K_\alpha = k_1 - p_\alpha k, \quad p_\alpha = \frac{A_k}{K_1} \tag{2.14}$$

The dot denotes differentiation with respect to time t. Equations (2.11) and (2.12) describe the motion of the centre of mass, and θ is the angle between the direction of one of the axes of the absolute system of coordinates. We can take as this axis the x axis of the right-handed system of coordinates (x, y) coinciding, for example, with the free surface of the medium, which occupies the half-space $y < 0$, and the direction of the velocity υ_m of the centre of mass. Equation (2.13) describes the angular motion about the centre of mass, while Eq. (2.14) is a consequence of the kinematic relation $\dot{\alpha} = \omega - \dot{\theta}$ and (2.12). In (2.13) $z_y^\mathrm{T} = z_y + 2\beta\mu/3$, which follows from (2.10) in the adopted approximation. I_z is the moment of inertia of the cone about the z axis, and M_0 and ρ_0 are the mass and mean density of the cone.

Using (2.4), (2.11) and (2.14), we change from Eq. (2.13) for $\dot{\omega}$ to the equation for k_1

$$\dot{k}_1 = -\frac{\upsilon_m}{L}\chi A_k\left[k_1 - \left(p_\omega + A_jk_1^2\right)k\right] \tag{2.15}$$

$$\chi = \frac{K_0}{I}, \quad A_j = \frac{\beta^2}{A_k\chi}$$

Taking into account that the parameter A in the local interaction model is of the same order as the density of the medium [1], in accordance with (2.11) we conclude that $A_k \sim 1$ for dense media. Moreover, since $I \sim 10^{-1}$, the term $A_jk_1^2$ can, in general, be discarded in (2.15) in the slender body approximation and for k, k_1 belonging to the domain (2.5). Later on we shall discuss the effect of this term onto the domain of variation of the governing parameters corresponding to stable solutions.

Therefore, it is necessary to find a solution of the autonomous system of equations (2.11), (2.12), (2.14) and (2.15) with initial data υ_{m0}, θ_0, ω_0, α_0. The initial conditions must ensure that the initial point belongs to the domain defined by (2.5) in the space of (k, k_1) and the parameters of the system (2.14), (2.15) must give rise to the corresponding trajectory in the same domain passing through the stationary point (0, 0) [4].

3. MOTION ABOUT THE CENTRE OF MASS. STABILITY ANALYSIS

The right-hand sides of (2.14) and (2.15) contain the same multiplier υ_m, which can be determined from (2.11) and can affect only the velocity of motion of a point along a trajectory in the phase space (k, k_1) [4]. It follows that within the framework of the local interaction model (1.1) with Coulomb friction the stability of the motion of the body in a dense medium when there is no flow separation can be investigated in the space (k, k_1) independently of the motion of the centre of mass. This is also true when the slender-body approximation is not employed. It suffices that the terms $D = C/[A\upsilon_m^2\sin^2\beta\cos^2(\mathrm{arctg}(k\ \mathrm{tg}\beta))]$ can be neglected in the corresponding time interval.

We introduce the variable

$$\eta = \frac{k_1}{k}, \quad \dot{\eta} \equiv \frac{1}{k}\left(\dot{k}_1 - \eta\dot{k}\right) = -\frac{v_m}{L} K_1 \zeta(\eta) \tag{3.1}$$

$$\zeta(\eta) = \eta^2 + p_\alpha(\chi - 1)\eta - \chi p_\alpha p_\omega$$

Changing in (2.11), (2.12), (2.14), (2.15) and (3.1) from t to the dimensionless distance s traversed by the centre of mass, using the relation $ds = v_m dt/L$, we obtain

$$\frac{dv_m^2}{ds} = -2v_m^2\beta^2 A_k\left\{\left(1+\frac{\mu}{\beta}\right)\left[1 + D + \frac{1}{2}K^2 + \frac{1}{36}k_1^2\right] + kK\right\} \tag{3.2}$$

$$\frac{d\theta}{ds} = \beta A_k K \tag{3.3}$$

$$\frac{dk}{ds} = K_1 K_\alpha, \quad \frac{dk_1}{ds} = -\chi A_k K_\omega \tag{3.4}$$

$$\frac{d\eta}{ds} = -K_1\zeta(\eta) \tag{3.5}$$

It is convenient to seek the phase-space trajectories of the system of linear homogeneous differential equations (3.4) in the (k, k_1) plane in the neighbourhood (2.5) of the singular stationary point (0, 0) in the parametric form

$$\frac{dk}{d\eta} = -\frac{k(\eta - p_\alpha)}{\zeta(\eta)}, \quad k_1 = k\eta \tag{3.6}$$

The zeros of the denominator in (3.6) and the roots $\lambda_{1,2}$ of the secular equation of system (3.4) are given by

$$\eta_{1,2} = p_\alpha\left[-(\chi - 1) \pm \sqrt{\Delta}\right]/2, \quad \Delta = (\chi - 1)^2 + 4\chi p_\omega / p_\alpha \tag{3.7}$$

$$\lambda_{1,2} = A_k\left[-(\chi + 1) \pm \sqrt{\Delta}\right]/2, \quad \lambda_1\lambda_2 = A_k^2\chi(1 - p_\omega / p_\alpha) \tag{3.8}$$

The type of singular point depends on the sign of the discriminant Δ, and for $\Delta > 0$ also on the signs of the roots of the secular equation and their product.

To fix our ideas, we shall consider a thin cone with a uniform mass distribution along l starting from the vertex. In this case

$$\frac{l}{L} = \frac{4}{3}\left(\frac{2}{3} - z_y\right), \quad I = \frac{1}{15}\left(\frac{2}{3} - z_y\right)^2$$
$$\chi = 15K_0\Big/\left(\frac{2}{3} - z_y\right)^2 > 1, \quad 0 < \frac{l}{L} \leqslant 1, \quad \frac{2}{3} > z_y \geqslant -\frac{1}{12} \tag{3.9}$$

Taking (2.13), (2.14) and (3.9) into account, the curve

$$A_k = 4z_y^\tau I\Big/\left[(K_0 - I)^2 + 4z_y z_y^\tau I\right] \tag{3.10}$$

turns out to be the curve corresponding to $\Delta = 0$ in (3.7) in the plane of the parameters (A_k, z_y).

By (2.13) and (2.14) $p_\alpha = p_\omega > 0$ for

$$z_y = -(A_k + 12\beta\mu)\Big/\left[18\left(1 - \frac{2}{3}\beta\mu A_k\right)\right] \equiv f(A_k) \tag{3.11}$$

The function (3.11) is represented by curve 1 in Fig. 1 Curve 2 in the same figure corresponds to $\Delta = 0$ (3.10) for $\beta\mu = 0.03$. Curve 3 corresponds to a discontinuity of the second kind of p_α in (2.14). The straight line $4 - z_y = -1/12$ corresponds to a homogeneous cone with $l = L$ in (3.9).

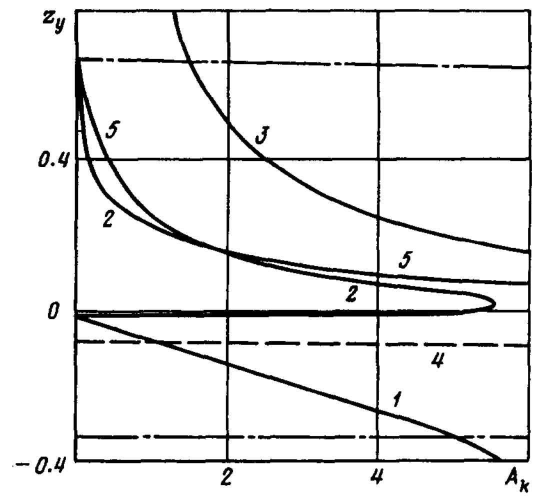

Fig. 1.

In the general case, for an arbitrary mass distribution inside the cone the only meaningful points (in Fig. 1) lie between the dash-dot lines, the upper one corresponding to $z_y = 2/3$ and the lower one to $-z_y = -1/3$. We also observe that the curve $\Delta = 0$ given by (3.10) must always lie to the left of curve 3, where p_α and p_ω have opposite signs.

The values $p_\alpha > p_\omega > 0$ and $\Delta > 0$ correspond to the points above curve 1 and below the straight line $z_y = -2\beta\mu/3$ ($z_y^T = 0, p_\omega = 0$) passing through the point of intersection of curve 1 and curve 2. According to [4], the stationary point (0, 0) in the phase plane (k, k_1) represents a stable node, since $\lambda_1\lambda_2 > 0$ and $\lambda_1 < 0$, $\lambda_2 < 0$ in (3.8). The trajectories of system (3.4) corresponding to this case are represented by the solid lines in Fig. 2(a). The directions of motion of points along the trajectories as s increases are indicated by arrows. The directions are defined by the signs of the first terms in equations (3.4), the zeros of which are represented by dashed straight lines with angular coefficients p_α and p_ω. The rectilinear trajectories $\eta = \eta_1$ and $\eta = \eta_2$ correspond to the zeros (3.7) of the denominator in (3.6). All trajectories, with the exception of the straight line $\eta = \eta_2$, are tangent at the stationary point on the straight line $k_1 = \eta_1 k$. For $z_y = -2\beta\mu/3$ (Fig. 1) we have $p_\omega = \eta_1 = 0$, since $\chi > 1$ in (3.9), and all trajectories, apart from $\eta = \eta_2$, will be tangent to the axis $k_1 = 0$.

In the range of values of (A_k, z_y) corresponding to the points above the straight line $z_y = -2\beta\mu/3$, between curves 2 and 3 (Fig. 1), we have $p_\alpha > 0$, $p_\omega < 0$. The stationary point is a stable node (Fig. 2b). As $1 - z_y A_k \to 0$ ($p_\alpha \to +\infty$) the type of singular point does not change and the rectilinear trajectories $\eta = \eta_2$ coincide with the axis $k = 0$ (Fig. 2c). In the region of values of (A_k, z_y) corresponding to the points above curve 3 (Fig. 1) we have $p_\alpha < p_\omega < 0$, $\eta_1 < 0$, $\eta_2 > 0$, $\lambda_1\lambda_2 > 0$, $\lambda_1 < 0$, $\lambda_2 < 0$. Therefore in these cases the trajectories form a stable node (Fig. 2d). If we let (A_k, z_y) in the domain between curves 2 and 3 tend to curve 2 ($\Delta = 0$), then the trajectories in the phase plane will take the form shown in Fig. 2(e), which is the limit form of the pattern of trajectories in Fig. 2(b) as $\eta_1 \to \eta_2$.

When $\Delta < 0$ in (3.7), which corresponds to points inside the domain bounded by curve 2 (Fig. 1), the stationary point is a focus with right-handed spiral-shaped trajectories (Fig. 2f).

If (A_k, z_y) (Fig. 1) tends from above to curve 1, corresponding to $p_\alpha = p_\omega$ as in (3.11), then we pass from the pattern of phase space trajectories presented in Fig. 2(a) to the degenerate case (Fig. 2g) in which the curve $k_1 = p_\alpha k$ consists of stationary points, that follows from Eqs (3.4). The trajectories are straight lines satisfying the differential equation $dk_1/dk = -\chi p_\alpha$, parallel to the direction of η_2 in (3.7) and leading to the stationary line. If (A_k, z_y) lies below curve 1 (Fig. 1), then $p_\omega > p_\alpha > 0$ and $\lambda_1\lambda_2 < 0$ in (3.8). The stationary point for system (3.4) is a saddle. Only two trajectories, namely, the straight lines with coefficient η_2 (Fig. 2h), lead towards it in directions opposite to one another.

Integrating Eqs (3.5) and (3.6), we find

$$\Delta > 0, \quad s - s_0 = -\frac{1}{A_k\sqrt{\Delta}} \ln\frac{u}{u_0} \tag{3.12}$$

$$k = k_0\left(\frac{u}{u_0}\right)^{\gamma} \frac{(1-u)}{(1-u_0)}$$

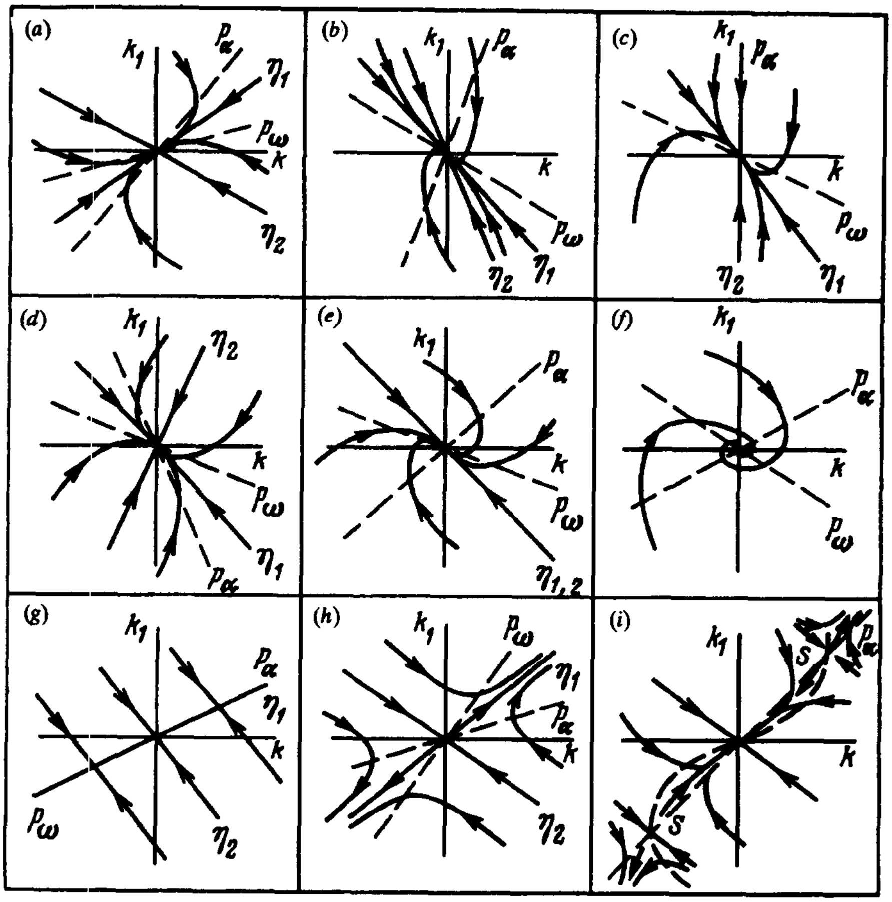

Fig. 2.

$$r = r_0\left(\frac{u}{u_0}\right)^{\gamma}\sqrt{\frac{(1-u)^2+(\eta_1-u\eta_2)^2}{(1-u_0)^2+(\eta_1-u_0\eta_2)^2}},\quad r=\sqrt{k^2+k_1^2}$$

$$u=\frac{\eta-\eta_1}{\eta-\eta_2},\quad \gamma=\frac{1}{2}\left(\frac{\chi+1}{\sqrt{\Delta}}-1\right)$$

$$\Delta=0,\quad s-s_0=\frac{\xi}{K_1},\quad \xi=\frac{\eta_0-\eta}{(\eta-\eta_1)(\eta_0-\eta_1)} \tag{3.13}$$

$$k=k_0\frac{\eta_0-\eta_1}{\eta-\eta_1}\exp(-\gamma_1\xi)$$

$$r=r_0\left|\frac{\eta_0-\eta_1}{\eta-\eta_1}\right|\sqrt{\frac{1+\eta^2}{1+\eta_0^2}}\exp(-\gamma_1\xi),\quad \gamma_1=\frac{1}{2}(\chi+1)p_\alpha$$

$$\Delta<0,\quad s-s_0=-\frac{2}{A_k\sqrt{-\Delta}}\quad(\varphi-\varphi_0) \tag{3.14}$$

$$k=k_0\frac{\cos\varphi}{\cos\varphi_0}\exp[\gamma_2(\varphi-\varphi_0)],\quad \gamma_2=\frac{\chi+1}{\sqrt{-\Delta}}$$

$$r = r_0 \sqrt{\frac{\Phi(\varphi)}{\Phi(\varphi_0)}} \exp\left[\gamma_2(\varphi - \varphi_0)\right]$$

$$\Phi(\varphi) = \cos^2\varphi + \frac{1}{4} p_\alpha^2 \left(\sqrt{-\Delta}\sin\varphi - (\chi - 1)\cos\varphi\right)^2, \quad \eta = \frac{1}{2} p_\alpha \left[\sqrt{-\Delta}\,\mathrm{tg}\,\varphi - (\chi - 1)\right]$$

The variables with zero subscript correspond to the initial state.

It is not difficult to verify that all the types of trajectories in the neighbourhood of the stationary point (0, 0) in the phase plane (k, k_1) qualitatively described above (Fig. 2) are contained in the resulting solution. In particular, the passage from a nodal point to a saddle point in (3.12) occurs when the sign of γ changes from plus to minus. The variable u varies in accordance with its dependence on η and the direction of motion of a point along a trajectory depending on the values of p_α and p_ω, the initial point (k_0, η_0) and the position of the ray η_0 relative to the directions η_1, η_2. Nevertheless, in all cases $|u| \to 0$ as $\eta \to \eta_1$.

We will now consider the effect of the term with coefficient A_j on the right-hand side of (2.15), which has not been taken into account in the above analysis. Without studying the curves corresponding to the zeros of the expression in square brackets in (2.15), we will only indicate that, apart from (0, 0) for $A_j \neq 0$, the points with coordinates

$$k = \pm\frac{1}{p_\alpha}\sqrt{\frac{p_\alpha - p_\omega}{A_j}}, \quad k_1 = \pm\sqrt{\frac{p_\alpha - p_\omega}{A_j}} \tag{3.15}$$

are the common zeros of the right-hand sides in (2.14) and (2.15). As can be seen from (3.15), the system of differential equations (2.14) and (2.15) has two more singular points if $p_\alpha > p_\omega$, which, according to Fig. 2, can only happen when $p_\alpha > 0$. It is easy to verify that these are saddle points. Trajectories corresponding to the case $p_\alpha > p_\omega > 0$ with two saddle points S are qualitatively presented in Fig. 2(i). These singular points must be taken into account in cases when the governing parameters and initial conditions allow them to fall into the domain (2.5) of flow past the cone without separation.

We shall find a condition under which the saddle points S (Fig. 2i) do not belong to the domain (2.5), which we define by the stronger inequalities

$$|k| < 1, \quad |k_1| < 2 \tag{3.16}$$

for simplicity. The expression under the root sign in (3.15) can be represented as

$$\frac{p_\alpha - p_\omega}{A_j} = p_\alpha Q \tag{3.17}$$

$$Q = \left[z_y\left(1 - \frac{2}{3}\beta\mu A_k\right) + \frac{1}{18}A_k + \frac{2}{3}\beta\mu\right]\left(I\beta^2\right)^{-1}$$

Obviously, in order that the points (3.15) should not fall into the domain (3.16) it suffices to require that the following system of inequalities is satisfied for $p_\alpha \geqslant 2$ in (2.14) and $p_\alpha < 2$, respectively

$$p_\alpha \geqslant 2, \qquad p_\alpha Q > 4$$
$$p_\alpha < 2, \qquad Q/p_\alpha > 1$$

Both systems of inequalities can be replaced by the stronger condition $Q > 2$, which we can write in the form

$$z_y > f(A_k) + 2I\beta^2 / (1 - \tfrac{2}{3}\beta\mu A_k) \tag{3.18}$$

taking (3.17) and (3.11) into account.

Using the expressions for I in (3.9) and $f(A_k)$ in (3.11), we can conclude that for $A_k \sim 1$, which corresponds to a wide range of ratios of the density of the medium and the mean density of the body, the second term in (3.18) is negligibly small. It only becomes important for small values of A_k. Thus, in the case of a cone moving in the air, when $A_k \sim 10^{-3}$ and $\mu = 0$, the second term in (3.18) has at least the same order as $f(A_k)$ and must be taken into account.

The analysis carried out above implies the important qualitative result that for the motion of a slender cone without separation in a dense medium, which includes various types of soil, within the framework of the local interaction model (1.1) the requirement for overall static stability is much relaxed, while for stable motion in the air a positive overall static stability (3.18) is needed, in general.

Thus, for $D_0 \ll 1$ and D 1 the motion of a slender cone in a dense medium will certainly be stable if the governing parameters ensure that the point (A_k, z_y) in the plane (Fig. 1) lies above curve 1, (3.18), and the whole trajectory in the phase plane (k, k_1) lies in the domain (2.5) of flow without separation. The motion of a body about the centre of mass for $D > 1$ will be discussed below.

4. THE MOTION OF THE CENTRE OF MASS

In Section 2 we assumed that $D_0 \ll 1$ and $D \leqslant 1$ in some time interval. This enabled us to ignore the terms in (2.12)–(2.14) of order $O(\beta^2 D)$ and $O(\beta\mu D)$ compared to unity and to obtain solution (3.12)–(3.14) for the motion of a body about the centre of mass. In the same characteristic time interval the general solution of Eq. (3.3) for the angle θ governing the direction of the velocity of the centre of mass in the stationary system of coordinates can be written using (3.5), (3.12)–(3.14) as follows:

$$\Delta > 0, \quad \theta - \theta_0 = \frac{k_0\beta}{(1-u_0)\sqrt{\Delta}}\left\{\frac{1}{\gamma}K_2(\eta_1)\left[1-\left(\frac{u}{u_0}\right)^{\gamma}\right] - \right.$$

$$\left. - \frac{u_0}{(1+\gamma)}K_2(\eta_2)\left[1-\left(\frac{u}{u_0}\right)^{1+\gamma}\right]\right\}, \quad K_2(\eta) = 1 + z_y\eta \tag{4.1}$$

$$\Delta = 0, \quad \theta - \theta_0 = \frac{2k_0\beta}{(1+\chi)}\{K_2(\eta_0)[1-\exp(-\gamma_1\xi)] +$$

$$\left. + \frac{1}{\gamma_1}K_2(\eta_1)(\eta_0 - \eta_1)[1-(1+\gamma_1\xi)\exp(-\gamma_1\xi)]\right\} \tag{4.2}$$

$$\Delta < 0, \quad \theta - \theta_0 = -\frac{k_0\beta}{2\chi\cos\varphi_0}\frac{\sqrt{-\Delta}\,p_\alpha}{(p_\alpha - p_\omega)}\{\Phi_1(\varphi) - \Phi_1(\varphi_0)\} \tag{4.3}$$

$$\Phi_1(\varphi) = \left\{K_2(p_\alpha)\sin\varphi + \gamma_2[K_2(p_\alpha) - \frac{2\chi}{1+\chi}z_y(p_\alpha - p_\omega)]\cos\varphi\right\}\exp[\gamma_2(\varphi - \varphi_0)]$$

The variables u, ξ and φ in (4.1)–(4.3) are defined in (3.12)–(3.14).

Note that by (3.3) the trajectory of motion of the centre of mass will be a straight line: $\theta = \theta_0$ if $\eta = -1/z_y$. This can be so only in those cases when the corresponding ray coincides with the rectilinear trajectories in the phase plane, i.e. $\eta_{1,2} = -1/z_y$ $(\Delta \geqslant 0)$. An analysis indicates that this type of motion of the centre of mass will occur in the case of a cone with a special mass distribution over the volume (3.9) for

$$A_k = 6\left(\frac{2}{3} - z_y\right)^2 / [5(1-12\beta\mu z_y)z_y]$$

The corresponding function is represented by the segments of curve 5 in Fig. 1, just like curve 2, for $\beta\mu = 0.03$. On the segment issuing from (0, 2/3) the rectilinear trajectory of the centre of mass will be realized for $\eta_0 = \eta_2 = -1/z_y$. On the second segment of curve 5 it will be realized for $\eta_0 = \eta_1 = -1/z_y$.

We consider Eq. (3.2) for the modulus of the velocity of the centre of mass. Suppose that $D \ll 1$ in some time interval. Then the general solution of Eq. (3.2), taking (3.5) and (3.12)–(3.14) into account, has the form

$$\Delta > 0, \quad v_m = v_{m0}\left(\frac{u}{u_0}\right)^{\gamma_3}\exp\left\{\frac{\gamma_3 k_0^2}{(1-u_0)^2}[R_1(u) - R_1(u_0)]\right\} \tag{4.4}$$

$$R_1(u) = \left[\frac{a_1}{\gamma} - \frac{1}{1+2\gamma}(a_2 + a_3)\left(u - \frac{1+2\gamma}{2\gamma}\right) + \frac{1}{2}a_3(u-1)^2\right]\left(\frac{u}{u_0}\right)^{2\gamma}$$

$$a_1 = \frac{1}{4}\Delta p_\alpha^2 K_0, \quad a_2 = \sqrt{\Delta}\, p_\alpha(a_4 + \eta_2 K_0), \quad a_4 = \left(1 + \frac{\beta}{\beta+\mu}\right)z_y$$

$$a_3 = \frac{1}{1+\gamma}\left\{\left[\frac{\beta}{\beta+\mu} + \frac{1}{2}K_2(\eta_2)\right]K_2(\eta_2) + \frac{1}{36}\eta_2^2\right\}, \quad \gamma_3 = \frac{\beta(\beta+\mu)}{\sqrt{\Delta}}$$

$$\Delta = 0, \quad v_m = v_{m0}\exp\left\{\beta(\beta+\mu)\, p_\alpha\left[\frac{k_0^2}{2\gamma_1}(R_2(\xi) - R_2(0)) - \xi\right]\right\} \tag{4.5}$$

$$R_2(\xi) = \left\{\frac{1}{4\gamma_1^2}(\eta_0 - \eta_1)^2\left(\frac{1}{2}K_0\eta_1^2 + a_4\eta_1 + a_5\right)[1 + (1 + 2\gamma_1\xi)^2] + \right.$$

$$+ \frac{1}{2\gamma_1}(\eta_0 - \eta_1)[K_0\eta_0\eta_1 + a_4(\eta_0 + \eta_1) + 2a_5](1 + 2\gamma_1\xi) +$$

$$\left. + \left(\frac{1}{2}K_0\eta_0 + a_4\right)\eta_0 + a_5\right\}\exp(-2\gamma_1\xi), \quad a_5 = \frac{a_4}{z_y} - \frac{1}{2}$$

$$\Delta < 0, \quad v_m = v_{m0}\exp\left\{\frac{2\beta(\beta+\mu)}{\sqrt{-\Delta}}\left[\varphi - \varphi_0 + \frac{k_0^2}{4\cos^2\varphi_0}(R_3(\varphi) - R_3(\varphi_0))\right]\right\} \tag{4.6}$$

$$R_3(\varphi) = \left\{\frac{a_6}{\gamma_2} - \frac{\Delta}{(1+\gamma_2^2)}[a_7(\gamma_2\cos 2\varphi + \sin 2\varphi) + \right.$$

$$\left. + a_8(\gamma_2\sin 2\varphi - \cos 2\varphi)]\right\}\exp[2\gamma_2(\varphi - \varphi_0)]$$

$$a_6 = a_5 + \frac{1}{2}(1-\chi)\, p_\alpha a_4 + \frac{1}{2}a_1\left[\frac{(1-\chi)^2}{\Delta} - 1\right]$$

$$a_7 = a_1 + a_6, \quad a_8 = \sqrt{-\Delta}\left[\frac{(1-\chi)}{\Delta}a_1 + \frac{1}{2}p_\alpha a_4\right]$$

We shall determine $u = u_1$ ($\Delta > 0$) in (3.12) when D and the sum of the remaining variable terms in the square brackets in (3.2) have the same value. Using (3.12) and (4.4), we obtain

$$D_0 = \frac{1}{2}\frac{k_0^2}{(1-u_0)^2}\left(\frac{u_1}{u_0}\right)^{2(\gamma+\gamma_3)}\exp\left\{\frac{2\gamma_3 k_0^2}{(1-u_0)^2}[R_1(u) - R_1(u_0)]\right\}\times$$

$$\times\left\{[(1-u_1)K_2(\eta_2) + z_y p_\alpha\sqrt{\Delta}]^2 + \frac{1}{18}[p_\alpha\sqrt{\Delta} + \eta_2(1-u_1)]^2\right\} \tag{4.7}$$

From (4.7) we find that for $\gamma \gg \gamma_3$ or, which is the same thing, for

$$1 - \frac{p_\omega}{p_\alpha} \gg \frac{(\chi+1)}{\chi}\beta(\beta+\mu) \tag{4.8}$$

the order of the right-hand side of (4.7) is defined by the factor $(u_1/u_0)^{2\gamma}$, since $|\, u\, | \to 0$ and the order of magnitude of the argument of the exponent does not exceed γ_3. This has an important consequence. If the governing parameters of the problem are such that (A_k, z_y) does not belong to a small neighbourhood over the curve $z_y = f(A_k)$ given by (3.11) (curve 1 in Fig. 1) on which the degenerate case of motion about the centre of mass is realized (Fig. 2g), then $\upsilon_{m1}/\upsilon_{m0}$ remains of the order of unity until D and the sum of the remaining variable terms in the square brackets in (3.2) become equal. In the principal approximation we can find u_1 from (4.7) and the corresponding values of υ_{m1} and s_1 from (4.4) and (3.12). We obtain

$$\frac{u_1}{u_0}=\left(\frac{2D_0}{r_0^2}\right)^{1/(2\gamma)}$$

$$\frac{\upsilon_{m1}}{\upsilon_{m0}}=1+\frac{\beta(\beta+\mu)}{\sqrt{\Delta}}\ln\left(\frac{u_1}{u_0}\right)[1+O[\beta(\beta+\mu)\ln D_0]] \tag{4.9}$$

$$s_1-s_0=-\frac{1}{A_k\sqrt{\Delta}}\ln\left(\frac{u_1}{u_0}\right)$$

Hence if condition (4.8) is satisfied, then for $u < u_1$ in (4.9) the terms on the right-hand side of (3.2), related to the motion of the body about the centre of mass, make a contribution to slowing down the body, the order of magnitude of which does not exceed $O(D_0)$, so they can be discarded. Starting from the given instant of time the velocity of the body can be determined from the equation

$$\frac{d\upsilon_m^2}{ds}=-2\upsilon_m^2\beta(\beta+\mu)A_k(1+D) \tag{4.10}$$

with the initial condition $\upsilon_m = \upsilon_{m1}$ determined from (4.9).

By a similar argument using (3.13), (4.5) and (3.14), (4.6) we can find ξ_1 and φ_1, respectively

$$\Delta=0,\quad \xi_1=-\frac{1}{2\gamma_1}\ln\frac{2D_0}{r_0^2} \tag{4.11}$$

$$\frac{\upsilon_{m1}}{\upsilon_{m0}}=1-\beta(\beta+\mu)p_\alpha\xi_1$$

$$s_1-s_0=-\frac{1}{A_k(1+\chi)}\ln\frac{2D_0}{r_0^2}$$

$$\Delta<0,\quad \varphi_1-\varphi_0=\frac{1}{2\gamma_2}\ln\frac{2D_0}{r_0^2} \tag{4.12}$$

$$\frac{\upsilon_{m1}}{\upsilon_{m0}}=1+\frac{2\beta(\beta+\mu)}{\sqrt{-\Delta}}(\varphi_1-\varphi_0)$$

In the case $\Delta < 0$ (4.12) the expression for $s_1 - s_0$ has the same form as for $\Delta = 0$ (4.11).

Thus, it has been shown that for $s \geqslant s_1$ the velocity of the centre of mass satisfies differential equation (4.10) with the assumed accuracy. The solution of this equation can be written as

$$s-s_1=\frac{1}{2\beta(\beta+\mu)A_k}\ln\left[\left(1+\frac{1}{D_0}\left(\frac{\upsilon_{m1}}{\upsilon_{m0}}\right)^2\right)\Big/\left(1+\frac{1}{D_0}\left(\frac{\upsilon_m}{\upsilon_{m0}}\right)^2\right)\right] \tag{4.13}$$

Whereas solution (4.4)–(4.6) for υ_m holds by (4.9), (4.11) and (4.12) when $0 < s - s_0 \leqslant s_1 - s_0 \sim -\ln D_0$, the solutions (3.12)–(3.14) for k and r and (4.1)–(4.3) for θ have the necessary accuracy when $D \leqslant 1$. We also observe that $r_1 \sim r_0\sqrt{D_0}$ when $s_1 - s_0 \sim -\ln D_0$.

We shall find the order of r for $D = 1$, which is equivalent to $(\upsilon_{m2}/\upsilon_{m0})^2 = D_0$. From (4.13) we can determine the distance travelled by the centre of mass during the time in which the velocity decreases from υ_{m1} to υ_{m2}. We have

$$s_2-s_1=\frac{1}{2\beta(\beta+\mu)A_k}\ln\left[\frac{1}{2}\left[1+\frac{1}{D_0}\left(\frac{\upsilon_{m1}}{\upsilon_{m0}}\right)^2\right]\right] \tag{4.14}$$

Using (4.9), (4.11), (4.12) and (4.14), we obtain an estimate for $s_2 - s_0$

$$\Delta > 0,\quad s_2 - s_0 = -\frac{1}{(1+\chi-\sqrt{\Delta})A_k}\ln\frac{2D_0}{r_0^2} + \frac{1}{2\beta(\beta+\mu)A_k}\ln\left[\frac{1}{2}\left[1+\frac{1}{D_0}\left(\frac{v_{ml}}{v_{m0}}\right)^2\right]\right] \sim -\frac{\ln D_0}{2\beta(\beta+\mu)A_k} \tag{4.15}$$

$$\Delta \leqslant 0,\quad s_2 - s_0 = -\frac{1}{(1+\chi)A_k}\ln\frac{2D_0}{r_0^2} + s_2 - s_1 \sim -\frac{\ln D_0}{2\beta(\beta+\mu)A_k}$$

Taking (4.15) into account, for $s = s_2$ we find an estimate of the distance r in (3.12)–(3.14) between a point on the phase space trajectory and the stationary point

$$\Delta > 0,\quad r \sim r_0(D_0)^{\gamma_4},\quad \gamma_4 = \frac{\gamma}{2\gamma_3} \tag{4.16}$$

$$\Delta \leqslant 0,\quad r \sim r_0(D_0)^{\gamma_5},\quad \gamma_5 = \frac{1+\chi}{4\beta(\beta+\mu)}$$

By (4.16) and (4.8) one can conclude that the motion about the centre of mass will have practically ceased when $s = s_2$. However, as the body is further slowed down, it may turn out that the terms containing D which were discarded in (2.12)–(2.15) tend to infinity faster than k and k_1 tend to zero. Under such conditions the right-hand sides of the equations of motion about the centre of mass (3.4) will increase without limit, which leads to the destruction of the "steady" motion at the final stage of slowing down, $v_m/v_{m0} \leqslant \sqrt{D_0}$. The term in the local interaction model that characterizes the deformation resistance capacity of the medium will be responsible for this.

The analysis shows that when relations (3.12), (4.9), (4.13), (4.14) and (4.16) are taken into account, destruction of motion can only occur if the condition

$$1 - \frac{p_\omega}{p_\alpha} \leqslant \frac{\chi+1}{\chi}\beta\ (\beta+\mu)$$

is satisfied. It follows that destruction of motion can only occur in a small neighbourhood of the degenerate case of motion (Fig. 2g), which corresponds to curve 1 in Fig. 1, and it can have two characteristic types. The first one occurs when the motion about the centre of mass will begin to "grow" after having ceased, and the second one, when, having approached some asymptotes, k and k_1 (Fig. 2g) will begin to move away from them for $D > 1$.

Substituting $s_2 - s_0$ from (4.15) into relations (4.1)–(4.3) for θ and using (3.12)–(3.14), we can verify that the terms responsible for the variation of the angle of inclination of the trajectory are beyond the order. It follows that for $s > s_2$ the centre of mass of the cone moves along a straight line whose angle of inclination is given by

$$\Delta > 0,\quad \theta - \theta_0 = \frac{k_0\beta}{(1-u_0)\sqrt{\Delta}}\left[\frac{1}{\gamma}K_2(\eta_1) - \frac{u_0}{1+\gamma}K_2(\eta_2)\right] \tag{4.17}$$

$$\Delta = 0,\quad \theta - \theta_0 = \frac{2k_0\beta}{1+\chi}\left[K_2(\eta_0) + \frac{1}{\gamma_1}(\eta_0 - \eta_1)K_2(\eta_1)\right]$$

$$\Delta < 0,\quad \theta - \theta_0 = \frac{k_0\beta\sqrt{-\Delta}\,p_\alpha}{2\chi\cos\varphi_0(p_\alpha - p_\omega)}\left\{(\sin\varphi_0 + \gamma_2\cos\varphi_0)K_2(p_\alpha) - \frac{2\gamma_2\chi}{1+\chi}z_y(p_\alpha - p_\omega)\cos\varphi_0\right\}$$

5. CONCLUSIONS AND REMARKS

Summarizing the above investigation of the dynamics of a slender rigid body of revolution with a weakly varying inclination of the longitudinal contour (cone) in a dense medium, given that there is no flow separation, we should first observe that the asymptotic behaviour of a slender body within the framework of the local interaction model, taking into account the effect of angular velocity of rotation and Coulomb friction with a dynamic pressure component on the contact surface, significantly exceeding the deformation resistance capacity of the medium at the initial instant of time enabled us to reduce the problem to a separate analysis of the motion of the body about the centre of mass in special phase-space variables, and on this basis, to an analysis of the motion of the centre of mass.

It is important that within the framework of the given interaction model the domain of variation of the governing parameters with asymptotically stable motion of the body contains a subdomain with negative values of the overall static stability, which is not the case for motion in a gas. In particular, a statically unstable slender cone with uniform mass distribution over its volume ($z_y = -1/12$, straight line 4 in Fig. 1) can move in a stable way in a dense medium when $A_k \geqslant 1$ in (2.11).

The results obtained cast doubt on the existing belief that the motion of a slender body in a dense medium is unstable, which is based on experimental data which depend at least on the following factors: the conditions under which the body approaches the boundary of a dense medium, which define the initial parameters of motion when the body is fully immersed in the medium, the presence of isotropy in the medium and the tensile strength of the body. Finally, according to the analysis carried out in Section 4, disorganization of the body motion, which is in fact asymptotically stable, can occur at the final stage if kD and (or) $k_1D \to \infty$ as $\upsilon_m \to 0$.

The results obtained enabled us to establish that the stable motion of a slender body can be separated into three characteristic stages. The first stage, which we shall call the transient regime, occurs when $s_0 \leqslant s \leqslant s_1$ ($s_1 - s_0 \sim -\ln D_0$), the velocity of the body varies only slightly, the motion about the centre of mass is essentially damped ($1 \geqslant r/r_0 \geqslant \sqrt{D_0}$), and the principal angle of inclination of the trajectory of the centre of mass ceases to vary ($\beta \geqslant |d\theta/ds| \geqslant \beta\sqrt{D_0}$).

In the second stage, called the regular regime, which occurs when $s_1 \leqslant s \leqslant s_2$, ($s_2 - s_1 \sim -\ln D_0/[2\beta(\beta + \mu)A_k]$) the body covers the longest distance, its motion about the centre of mass subsides ($\sqrt{D_0} \geqslant r/r_0 \geqslant 0$), and the angle of inclination of the trajectory approaches the asymptote ($\beta\sqrt{D_0} \geqslant |d\theta/ds| \geqslant 0$).

During the third and final stage, called the rectilinear regime, which occurs when $s_2 \leqslant s \leqslant s_3$ ($s_3 - s_2 \sim \ln 2/[2\beta(\beta + \mu)A_k]$) when the deformation resistance capacity of the medium exceeds the dynamic component in the interaction model, the body covers the second longest distance without moving about the centre of mass, moving along a rectilinear trajectory until it comes to rest or enters a terminal stage in which the stable motion is disrupted.

The trajectory of the centre of mass is given by

$$x = x_0 + L\int_{s_0}^{s} \cos\theta\, ds, \quad y = y_0 + L\int_{s_0}^{s} \sin\theta\, ds$$

The solution obtained can be used both in the case of stable and unstable trajectories in the phase plane (Fig. 2) as long as (k, k_1) lies in the domain (2.5) of flow without separation.

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